Boundary-layer flow at a saddle point of attachment

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This paper is a study of the flow of a viscous incompressible fluid in the immediate neighbourhood of a saddle point of attachment, near which the external flow is irrotational with components $\{ax, by, -(a+b)z\}$, where $a > 0 > b$. It is shown that the flow is of a boundary-layer character, and that part of the boundary-layer flow is reversed when $b/a < -0{\cdot}4294$.

On the assumption that such flows are physically plausible, the problem may be solved for all values of $b/a \geqslant -1$. Even in the limiting case $b/a = -1$, an effect of the boundary layer is everywhere to draw fluid towards the wall, so that vorticity is still convected towards the wall.

Numerical solutions have been computed, and some of the results are presented in the tables and diagrams.

1. Introduction

When a viscous fluid flows past a body which is simply connected and has a smooth surface, the fluid velocity at a small distance z from the surface is $\boldsymbol{\epsilon}z + O(z^2)$, where $\boldsymbol{\epsilon}$ is tangential to the surface S of the body. In fact $\mu\boldsymbol{\epsilon}$ denotes the skin-friction vector, that is the tangential component of the stress acting on S, where μ is the coefficient of viscosity. A point of the surface at which $\boldsymbol{\epsilon} = 0$ is a stagnation point of the flow and is a 'singular' point of the system of skin friction and vortex lines on S. Such points may be classified into two types, nodal points and saddle points. Moreover, these points are said to be points of attachment or separation according as the fluid near the stagnation streamline is moving towards or away from the surface.

The range of possible patterns of skin friction and vortex lines on such a surface is subject to a topological law, namely, that the number of nodal points must exceed the number of saddle points by 2. A proof of this theorem is given in the Appendix to this paper, together with definitions of the different types of singular points.

It suffices to say here that for a stagnation point of attachment P of a streamline from far upstream, where the flow is irrotational, we may express $\boldsymbol{\epsilon}$ in the form (ax, by), where $P(x, y)$ is the tangent plane at P and x, y are Cartesian co-ordinates. Points P such that both a and b are positive are nodal points of attachment, points P such that one (and only one) of a, b is negative are saddle points of attachment, provided $a+b > 0$. Saddle points of the flow may in some cases be related to the geometrical saddle points of the surface of the body, as will be discussed later. In general boundary-layer theory is not applicable at stagnation points of separation, since at these points vorticity is not convected towards the body's surface.

In the neighbourhood of a stagnation point of attachment, however, the flow of a viscous fluid is of a boundary-layer character even at moderate Reynolds numbers. Solutions for the flow at two-dimensional and axi-symmetrical stagnation points of attachment indicate that the additional terms in the full Navier–Stokes equations are identically zero, provided that the curvature of the body is so small that the surface may be taken to be plane. For the flow at a three-dimensional nodal point of attachment (Howarth 1951) it is also known that the boundary-layer equations yield similarly solutions which are full solutions of the Navier–Stokes equations.

The work of Howarth is important here in that he shows that with a suitable choice of orthogonal axes (x, y, z) the mathematical solution is the same as if the surface were plane, with the x- and y-axes in the plane, and with the external flow given by $\{ax, by, -(a+b)z\}$, where a and b are positive constants.

The purpose of this paper is to discuss the flow near a stagnation point which is a saddle point of attachment. Here again the solution, with suitable choice of axes, is the same as for the problem with a plane surface, in which lie the x and y axes, with the external flow given by $\{ax, by, -(a+b)z\}$, but where now $-a \leqslant b < 0$. The case $b < -a$ would correspond to a saddle point of separation, but it will be shown in §5 that the equations cannot then be solved.

Saddle points of attachment will occur on suitably shaped bodies placed in a uniform stream or, more practically, may occur for instance between the wing of an aeroplane and an engine mounting, the flow at which may affect separation of the laminary boundary layer over a larger area of the wing.

The present work and that of Howarth only holds near stagnation points for which the external flow is irrotational. It is hoped to extend the treatment in a later paper to discuss cases when the external flow possesses vorticity.

2. The flow past a wavy cylinder

As mentioned in the Introduction of this paper, it is possible for the inviscid flow at a stagnation point to be a saddle point of attachment. For an example of such a case let us consider the flow of an inviscid uniform stream past the cylinder

$$r = r_0(1+\epsilon\cos\lambda z),$$

as shown in figure 1. We choose ϵ very small and use z to measure distance parallel to the axis of the cylinder, the uniform stream being perpendicular to this axis.

The potential ϕ of the inviscid flow satisfies Laplace's equation $\nabla^2\phi = 0$. We seek a solution of the form $\phi = \phi_0 + \epsilon\phi_1 + O(\epsilon^2)$, where ϕ_0 and ϕ_1 are independent of ϵ. Now ϕ_0 is the potential for flow past the cylinder $r = r_0$, so that

$$\phi_0 = -U\left\{r+\frac{r_0^2}{r}\right\}\cos\theta. \tag{2.1}$$

Now

$$\nabla^2\phi_1 = 0, \tag{2.2}$$

and on the wall we require

$$\frac{\partial\phi}{\partial r}\cos\gamma + \frac{\partial\phi}{\partial z}\sin\gamma = 0,$$

where γ is the angle between the normal to the surface and the radial direction, since the component of velocity normal to the wall vanishes.

Since $\gamma = O(\epsilon)$, we have

$$\frac{\partial \phi_0}{\partial r} + \epsilon \frac{\partial \phi_1}{\partial r} = O(\epsilon^2) \tag{2.3}$$

when $r = r_0(1+\epsilon \cos \lambda z)$. Thus

$$-U\{1-(1+\epsilon\cos\lambda z)^{-2}\}\cos\theta + \epsilon\frac{\partial \phi_1}{\partial r} = O(\epsilon^2),$$

whence

$$\left[\frac{\partial \phi_1}{\partial r}\right]_{r=r_0} = 2U\cos\lambda z\cos\theta. \tag{2.4}$$

Hence we seek a solution of (2.2) of the form $\phi_1 = \Phi(r)\cos\lambda z\cos\theta$, and we obtain

$$\phi_1 = \{AI_1(\lambda r) + BK_1(\lambda r)\}\cos\lambda z\cos\theta.$$

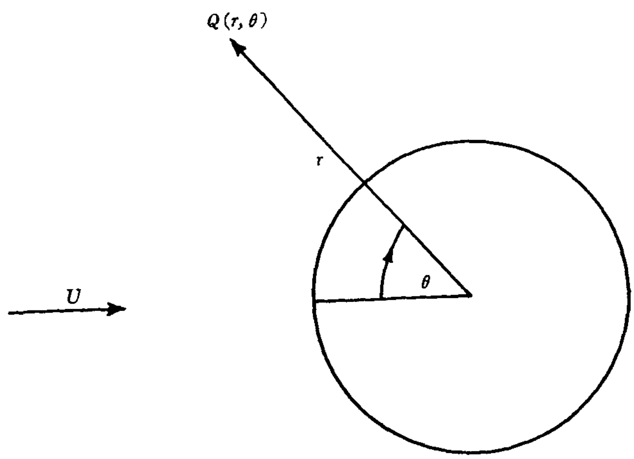

FIGURE 1. Flow past the cylinder $r = r_0(1+\epsilon\cos\lambda z)$.

Since $I_1(\lambda r) \sim \dfrac{e^{\lambda r}}{(2\pi\lambda r)^{\frac{1}{2}}}$ we take $A = 0$ so that the outer boundary condition may be satisfied. Then (2.4) gives $B = 2U/\{\lambda K_1'(\lambda r_0)\}$, so that

$$\phi_1 = \frac{2UK_1(\lambda r)}{\lambda K_1'(\lambda r_0)}\cos\lambda z\cos\theta.$$

Thus the velocity components (u_z, u_r, u_θ) are given by

$$u_z = -\frac{2U\epsilon K_1(\lambda r)}{K_1'(\lambda r_0)}\sin\lambda z\cos\theta,$$

$$u_r = -U\left\{1-\frac{r_0^2}{r^2}\right\}\cos\theta + \frac{2U\epsilon K_1'(\lambda r)}{K_1'(\lambda r_0)}\cos\lambda z\cos\theta,$$

$$u_\theta = +U\left\{1+\frac{r_0^2}{r^2}\right\}\sin\theta - \frac{2U\epsilon K_1(\lambda r)}{\lambda r K_1'(\lambda r_0)}\cos\lambda z\sin\theta.$$

Now set $z = \pi/\lambda + \hat{z}$ and $r = r_0(1-\epsilon) + \hat{r}$ where, for given ϵ, $\hat{z}$ and $\hat{r}$ are arbitrarily small, so that to the first order in $\hat{z}$ and $\hat{r}$ the flow near the geometrical saddle point $z = \pi/\lambda$, $r = r_0(1-\epsilon)$, $\theta = 0$ is given by

$$u_z = \frac{2U\lambda\epsilon K_1(\lambda r_0)}{K_1'(\lambda r_0)}\hat{z}, \tag{2.5}$$

$$u_r = -\left\{\frac{U}{r_0}(2+6\epsilon) + \frac{2U\lambda\epsilon K_1''(\lambda r_0)}{K_1'(\lambda r_0)}\right\}\hat{r}, \tag{2.6}$$

$$u_\theta = \left\{\frac{U}{r_0}(2+4\epsilon) + \frac{2U\epsilon K_1(\lambda r_0)}{\lambda r_0^2 K_1'(\lambda r_0)}\right\}\theta. \tag{2.7}$$

Since $K_1(\lambda r_0) > 0$ and $K_1'(\lambda r_0) < 0$ for $\lambda r_0 > 0$, these equations show that the flow has a saddle point of attachment at the geometrical saddle point

$$\hat{z} = \hat{r} = \theta = 0.$$

3. The flow near a singular point

Howarth (1951) obtained equations for the boundary-layer flow over a general curved surface S. He used a triply-orthogonal co-ordinate system (ξ, θ, ζ), such that ξ = constant and θ = constant represent developable surfaces formed by the normals to the lines of curvature of S, and ζ = constant represents a surface parallel to S.

In fact the restriction to lines of curvature is unnecessary, as Howarth's equations are valid for any orthogonal system of curves ξ = constant, θ = constant on S, as shown by Squire (1957).

Let us choose such an arbitrary system (ξ, θ, ζ) and suppose that the length elements are $h_1 d\xi$, $h_2 d\theta$, $h_3 d\zeta$ in the ξ, θ, ζ directions. Here h_3 is by definition a function of ζ alone, so that we may set $h_3 = 1$ and use ζ to measure distance from S.

If u, v, w are the velocity components in the ξ, θ, ζ directions, respectively, the boundary-layer equations (assuming that h_1, h_2 and all their first derivatives are not large) are as follows:

$$\frac{\partial u}{\partial t} + \frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial \zeta} - K_2 uv + K_1 v^2 = -\frac{1}{\rho h_1}\frac{\partial p}{\partial \xi} + \nu\frac{\partial^2 u}{\partial \zeta^2}, \tag{3.1}$$

$$\frac{\partial v}{\partial t} + \frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial \zeta} + K_2 u^2 - K_1 uv = -\frac{1}{\rho h_2}\frac{\partial p}{\partial \theta} + \nu\frac{\partial^2 v}{\partial \zeta^2}, \tag{3.2}$$

$$\frac{1}{h_1}\frac{\partial u}{\partial \xi} + \frac{1}{h_2}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \zeta} - K_1 u - K_2 v = 0. \tag{3.3}$$

Here $$K_1 = -\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi} \quad \text{and} \quad K_2 = -\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \theta}$$

are respectively the geodesic curvatures of the curves ξ = constant and θ = constant.

In equations (3.1), (3.2) and (3.3) the values of h_1, h_2, K_1 and K_2 may, to the orders of approximation involved, be taken in the surface S. The pressure p is supposed to be prescribed by the inviscid flow, as its variation across the boundary layer is negligible.

Now let P be a stagnation point of attachment of the flow over S. Choose the co-ordinates (ξ, θ, ζ) such that at P, $\xi = \theta = \zeta = 0$ and $h_1 = h_2 = 1$, and such that the boundary-layer equations hold in the form (3.1) to (3.3) near P.

If the mainstream has velocity components U, V, then, by irrotationality,

$$h_2 \frac{\partial V}{\partial \xi} - h_1 \frac{\partial U}{\partial \theta} + V \frac{\partial h_2}{\partial \xi} - U \frac{\partial h_1}{\partial \theta} = \theta, \tag{3.4}$$

so that near P, neglecting terms of $O(\xi, \theta)$,

$$\frac{\partial V}{\partial \xi} - \frac{\partial U}{\partial \theta} = 0. \tag{3.5}$$

Thus, if we choose our co-ordinate system (ξ, θ) on the surface so that near P, $U = A\xi$, then also $V = B\theta$, where A and B are constants. In equations (3.1) to (3.3), the terms of lowest order near P are obtained by neglecting the curvature terms and replacing h_1 and h_2 by 1. Moreover, to the same order of approximation, we may suppose the surface to be plane at P with ζ measured normal to the surface so that in the neighbourhood of P the ξ-, θ-, ζ-axes are rectangular.

We now re-name these axes—$P(\xi, \theta, \zeta)$ becoming $P(x, y, z)$—and set $A\xi = ax$ and $B\theta = by$. Hence, as might have been expected, we see that a first solution to the problem is the flow bounded by the plane $z = 0$ with an external velocity $\{ax, by, -(a+b)z\}$, where $P(x, y, z)$ are rectangular cartesian co-ordinates. We do not consider the case when a and b are both negative, which would be appropriate for a nodal point of separation. Hence one of a and b is positive and we shall take the greater of these to be $a > 0$.

Howarth (1951) showed that the velocity components (u, v, w) referred to the axes $P(x, y, z)$ may be taken in the form

$$u = axf'(\eta), \quad v = byg'(\eta), \quad w = -\nu^{\frac{1}{2}}\{af(\eta) + bg(\eta)\}/a^{\frac{1}{2}}, \tag{3.6}$$

where $\eta = a^{\frac{1}{2}}z/\nu^{\frac{1}{2}}$ is the dimensionless distance from the surface. Equation (3.3) is then satisfied, and equations (3.1) and (3.2) become

$$f''' + (f + cg)f'' + (1 - f'^2) = 0, \tag{3.7}$$

$$g''' + (f + cg)g'' + c(1 - g'^2) = 0, \tag{3.8}$$

where $c = b/a$.

The boundary conditions for (3.7) and (3.8) are

$$\left.\begin{array}{ll} f = g = f' = g' = 0 & \text{when} \quad \eta = 0, \\ f' \to 1, \; g' \to 1 & \text{as} \quad \eta \to \infty. \end{array}\right\} \tag{3.9}$$

The solution given above leads to a solution of the full Navier–Stokes equations of motion, as the variation of pressure across the boundary layer can be determined from the z-momentum equation. When $c = 1$ we have $f = g$, and this gives the flow at an axi-symmetrical stagnation point. When $c = 0$, we have $b = 0$ and the flow is that in the neighbourhood of a two-dimensional stagnation point, as on a cylinder in a uniform stream perpendicular to its axis.

For intermediate values of c the velocity profiles given by $f'(\eta)$ and $g'(\eta)$ are of a boundary-layer type, as shown by Howarth who, in his paper, calculated these profiles for $c = 0{\cdot}25$, $0{\cdot}50$ and $0{\cdot}75$, besides giving the already known solutions for $c = 0$ and $c = 1$.

4. The flow near a saddle point of attachment

In his paper Howarth (1951) stated that the solutions of $f'(\eta)$ and $g'(\eta)$ for $c < 0$ may be found from

$$f(\eta, -c) = f(\eta, c), \quad g(\eta, -c) = -g(\eta, c),$$

but these equations are incorrect since they do not satisfy the boundary condition at infinity. Thus solutions with $c < 0$ cannot be found from Howarth's results.

The component of velocity normal to the surface near the stagnation point is $\{-a(1+c)z\}$, which is negative when $c > -1$; in the corresponding viscous flows the vorticity is therefore being convected inwards at the edge of the boundary layer. Thus it is likely that the conditions at infinity can be satisfied only when $c > -1$, and that equations (3.7) and (3.8) will then have solutions of a boundary-layer form. Values of c between 0 and -1 correspond to saddle points of attachment.

A flow with $0 > c > -1$ can occur when an otherwise uniform stream flows past a suitably shaped body, as was shown in §2. It is desirable therefore to obtain quantitative results from equations (3.7) and (3.8) when $c < 0$, with the boundary conditions (3.9). These equations were integrated using the Manchester Mercury Computer. To solve the two-point boundary-value problem a special integration programme was written and solutions for $f(\eta)$ and $g(\eta)$ and their derivatives were obtained, for different negative values of c, correct to three decimal places.

For a certain range of integration the programme evaluated the dependent variables at specified subintervals of the range and an auxiliary routine was used to tabulate the results. The integration programme used the Runge–Kutta process with an error per step of order h^5, where h is a step length. To ensure a given accuracy in each subinterval of size d say, the results of using first p substeps and then $p+1$ substeps were compared. If these differed by less than a preassigned small quantity the machine proceeded to the next subinterval. Otherwise p was increased until agreement was obtained or until there was no further improvement.

For each value of the parameter c the corresponding values of $s = f''(0)$ and $t = g''(0)$ had to be determined so that when the equations were integrated outwards from the origin the solutions had the correct asymptotic behaviour.

It was found that for each specific value of c there existed a line of points (s, t) all of which gave solutions satisfying the outer boundary conditions. Thus the solution is not unique and it was found that the end-point of the line, which was semi-infinite in extent, was the one that made $f'(\eta)$ and $g'(\eta)$ approach unity from below more quickly than any other solution without either ever becoming equal to unity. These solutions (for different c) were taken to be the correct ones as they gave most rapid approach to the outer boundary condition, minimizing the boundary-layer thickness. These solutions are the ones whose asymptotes are derived solely from exponential terms.

Let us determine for large values of η the asymptotic solutions of equations (3.7) and (3.8).

Let
$$\alpha = \lim_{\eta\to\infty}(\eta - f), \quad \beta = \lim_{\eta\to\infty}(\eta - g),$$
and put $f = \eta - \alpha - H$, $g = \eta - \beta - K$. Then if
$$F = H'(\eta) = 1 - f' \quad \text{and} \quad G = K'(\eta) = 1 - g',$$
the result of linearizing (3.7) and (3.8) is
$$F'' - \{(\alpha + \beta c) - \eta(1+c)\}F' - 2F = 0, \tag{4.1}$$
$$G'' - \{(\alpha + \beta c) - \eta(1+c)\}G' - 2cG = 0, \tag{4.2}$$
where primes denote differentiation with respect to η.

Now choose a new independent variable
$$\chi = k\left\{\eta - \left(\frac{\alpha + \beta c}{1+c}\right)\right\} \quad (c > -1),$$
where $k = (1+c)^{\frac{1}{2}}$ which is > 0, and (4.1), (4.2) become respectively
$$F'' + \chi F' - \frac{2}{1+c}F = 0, \tag{4.3}$$
$$G'' + \chi G' - \frac{2c}{1+c}G = 0. \tag{4.4}$$

The asymptotic solutions of (4.3) and (4.4) then give
$$1 - f' \sim A_1 e^{-\frac{1}{2}\chi^2}\chi^{\{2/(1+c)+1\}}\{1 + o(1)\} + B_1\chi^{2/(1+c)}\{1 + o(1)\} \tag{4·5}$$
$$1 - g' \sim A_2 e^{-\frac{1}{2}\chi^2}\chi^{\{2c/(1+c)+1\}}\{1 + o(1)\} + B_2\chi^{2c/(1+c)}\{1 + o(1)\}. \tag{4.6}$$

The values of A_1, A_2, B_1, B_2 will depend upon the position of the point (s, t). In order to satisfy the boundary conditions at infinity, we must have $B_1 = 0$ but (since $-1 < c < 0$) B_2 is left arbitrary.

Thus there will be a curve in the (s, t)-plane given by $B_1 = 0$ and the solution which approaches its asymptote most rapidly will be obtained from that point (s^*, t^*) of the curve which makes $B_2 = 0$. The solutions which arise from points of the curve $B_1 = 0$ on one side of (s^*, t^*) will make $B_2 < 0$, and so g' would approach its limit from above, whereas those on the other side will make $B_2 > 0$, so that g' remains < 1. Thus, for each value of c the required point (s^*, t^*) is the intersection of the curves $B_1(s, t) = 0$ and $B_2(s, t) = 0$.

A special subprogramme was written for the Mercury Computer to determine the appropriate points (s^*, t^*) as accurately as possible. For each case s^* was determined to 8 places of decimals and t^* to 5 places.

When c is slightly negative the solutions for f and g resemble those with $c > 0$. As c decreases $g''(0)$ decreases so that the skin-friction along the y-axis is reduced as c becomes more negative. When $c = -0{\cdot}4294$ one finds that $g''(0) = 0$, indicating that there is a tendency for some of the flow to reverse its direction. When $c < -0{\cdot}4294$ reversal of the v-component of flow does in fact occur. However, the velocity profiles given by $f'(\eta)$ and $g'(\eta)$ are still of the same character as velocity profiles in two-dimensional boundary-layer flows with separation.

When $c = -1$ solutions satisfying the outer boundary condition are still obtainable, the amount of reversed flow is greatest and the greatest value of

$-g'(\eta)$, the backflow velocity, is roughly 0·33. For this case ($c = -1$) equations (4.1) and (4.2) take the form

$$F'' + (\beta - \alpha) F' - 2F = 0, \tag{4.7}$$

$$G'' + (\beta - \alpha) G' + 2G = 0. \tag{4.8}$$

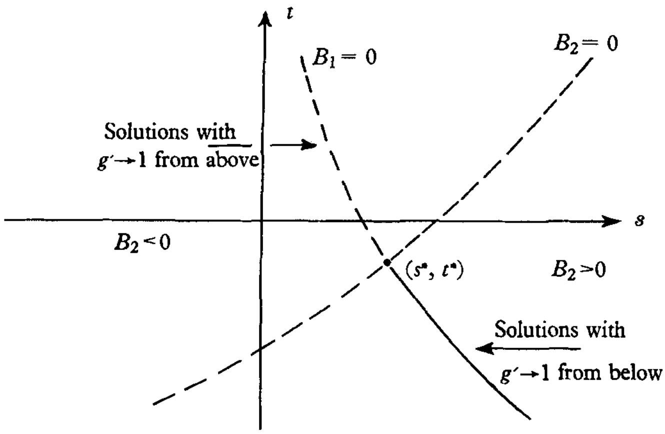

FIGURE 2. The (s, t)-plane.

Moreover, it was found from the numerical integration of (3.7) and (3.8) that when $c = -1$, $\beta - \alpha = 2\sqrt{2}$ to 5 decimals. Hence the asymptotic solutions are

$$1 - f' \sim A_1 e^{-(2+\sqrt{2})\eta}\{1 + o(1)\} + B_1 e^{(2-\sqrt{2})\eta}\{1 + o(1)\}, \tag{4.9}$$

$$1 - g' \sim A_2 e^{-\sqrt{2}\eta}\{1 + o(1)\} + B_2 \eta e^{-\sqrt{2}\eta}\{1 + o(1)\}, \tag{4.10}$$

and the appropriate ones, for the same reasons as before, are given by $B_1 = B_2 = 0$.

Thus even when $c = -1$ equations (3.7) and (3.8) still yield solutions which satisfy the Navier–Stokes equations for the flow over an infinite plane with an external flow $(ax, -ay, 0)$. The three-dimensional boundary-layer displacement thickness which we define, in accordance with Lighthill (1958), to be the thickness of that layer which when attached to the original surface modifies the inviscid flow outside it by the same amount as the boundary layer, is now infinite and for large values of η we have $w = -2(2a\nu)^{\frac{1}{2}}$. When η is small

$$w \sim -\nu^{\frac{1}{2}} a^{\frac{1}{2}} \{f''(0) - g''(0)\} \eta^2,$$

so that the negative value of $g''(0)$, which indicates a reversal of flow, contributes to the flow towards the surface and so opposes the spreading of vorticity away from the surface. Thus a boundary-layer type of flow is still possible.

Solutions of equations (3.7) and (3.8) with the boundary conditions (3.9) were obtained for thirteen different negative values of $c \geqslant -1$ and some of these results are given in §7, which contains the numerical results.

Finally we give the appropriate formulae for the various boundary-layer displacement thicknesses for general c. Since $w = -\nu^{\frac{1}{2}}(af + bg)/a^{\frac{1}{2}}$, we have for large η

$$w = -\nu^{\frac{1}{2}}\{a(\eta - \alpha) + b(\eta - \beta) + o(1)\}/a^{\frac{1}{2}},$$

where α and β are as defined earlier. Hence the three-dimensional boundary-layer displacement thickness δ_1 at the saddle point is given by

$$\delta_1 = \frac{\nu^{\frac{1}{2}}}{a^{\frac{1}{2}}}\left\{\frac{\alpha+c\beta}{1+c}\right\}. \tag{4.11}$$

The two-dimensional boundary-layer displacement thicknesses δ_x, δ_y in the x-, y-directions respectively are given by

$$\frac{a^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\delta_x = \int_0^\infty (1-f')\,d\eta = \lim_{\eta\to\infty}(\eta-f) = \alpha,$$

$$\frac{a^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\delta_y = \int_0^\infty (1-g')\,d\eta = \lim_{\eta\to\infty}(\eta-g) = \beta.$$

Some values of δ_1, δ_x, and δ_y, are tabulated for different values of c in table 1 of §7.

5. Insolubility of the case $c < -1$

When $c < -1$ the external flow has $w = -a(1+c)z$ so that $w > 0$. Hence in the corresponding viscous flow it seems likely that vorticity will beconvected away from the surface into the mainstream. Because of this we expect conditions at infinity to be upset, and we prove below that no solutions of (3.7) and (3.8) exist which satisfy all the boundary conditions (3.9). We restrict our attention to the existence of solutions f and g which possess derivatives of every order at all points in the range $0 \leqslant \eta \leqslant \infty$.

Lemma 1. No solution $g'(\eta)$ exists which has a stationary value of 1 for finite η.

Proof. Equation (3.8) may be re-arranged in the form

$$g''' = c(g'^2-1)-(f+cg)g''. \tag{5.1}$$

Suppose for $\eta = \eta_1$ we have $g'(\eta_1) = 1$ and $g''(\eta_1) = 0$. Then it follows from (5.1) and the derivatives of this equation that g''' and all higher derivatives are zero when $\eta = \eta_1$, hence $g' \equiv 1$. The boundary condition $g'(0) = 0$ is thus not satisfied and the Lemma is proved.

Lemma 2. When $c < 0$ and g' has a stationary value then if $|g'| < 1$ it is a minimum and if $|g'| > 1$ it must be a maximum.

Proof. If g' has a stationary value, then $g'' = 0$ at this point and $g''' = c(g'^2-1)$ by equation (5.1).

Thus if $c < 0$ and $|g'| < 1$, $g''' > 0$ so that g' is a minimum. If $c < 0$ and $|g'| > 1$, $g''' < 0$ and g' is a maximum.

Theorem 1. There is no solution $g'(\eta)$ when $c < -1$ such that $g' \to 1$ as $\eta \to \infty$, with $|g'| < 1$ for all $\eta > \eta_0$.

Proof. Suppose if possible that such a solution $g'(\eta)$ exists. By lemma 2, since $|g'| < 1$, g' has for $\eta > \eta_0$ at most one stationary value because one cannot have two consecutive stationary values which are both minima.

Also since $g' \to 1$ as $\eta \to \infty$ and $|g'| < 1$ for all $\eta > \eta_0$ then g'' must be positive for all sufficiently large η.

Moreover $f+cg \sim (1+c)\eta$ will, for $c < -1$, be negative for all sufficiently large η. Hence (5.1) gives $g''' > 0$ for all sufficiently large η. Since $g'' > 0$ this contradicts the assumption that $g' \to 1$ and so no such solutions exists.

Theorem 2. There is no solution $g'(\eta)$ when $c < -1$ such that $g' \to 1$ as $\eta \to \infty$, with $g' > 1$ for all $\eta > \eta_0$.

Proof. Similar to that given in theorem 1.

Lemma 3. If, when $c < -1$, $f+cg < 0$ for all $\eta > \eta_0$ and $g''(\eta)$ vanishes for $\eta = \eta_1, \eta_2, \ldots$ with $\eta_0 < \eta_1 < \eta_2 < \ldots$, then the sequence $g'(\eta_i)$ does not tend to 1 as $i \to \infty$.

Proof. Multiply (5.1) throughout by $g''(\eta)$ and integrate between η_r and η_s, where $g''(\eta_r) = g''(\eta_s) = 0$ with $\eta_0 < \eta_r < \eta_s$.

Then
$$[\tfrac{1}{2}g''^2]_{\eta_r}^{\eta_s} = \tfrac{1}{3}c[(g'^3-3g')]_{\eta_r}^{\eta_s} - \lambda, \tag{5.2}$$
where $\lambda = \int_{\eta_r}^{\eta_s} (f+cg)\, g''^2 d\eta$.

Since $f+cg < 0$ for all $\eta > \eta_0$, λ is negative and (5.2) gives
$$(g'^3-3g')_{\eta_s} > (g'^3-3g')_{\eta_r}.$$

As $g' = 1$ makes (g'^3-3g') a minimum when regarded as a function of g', we therefore cannot have $g'(\eta_i) \to 1$ as $i \to \infty$.

Theorem 3. Equation (5.1) or (3.8) has no solution satisfying $g'(\eta) \to 1$ as $\eta \to \infty$ when $c < -1$.

Proof. Suppose if possible such a solution $g'(\eta)$ exists. Then it follows from theorems 1 and 2 that g' must take values on both sides of 1 as $\eta \to \infty$. Hence there must be a distinct sequence η_i with $\eta_{i+1} > \eta_i$ and $\lim_{i\to\infty} \eta_i = \infty$ at which g' has stationary values. Since $f'+cg' \to 1+c < 0$ we can choose η_0 such that $f+cg < 0$ for all $\eta > \eta_0$. Then by lemma 3 the sequence $g'(\eta_i)$ does not tend to 1, contrary to hypothesis.

We might note that theorem 3 disproves the existence of mathematical solutions of equations (3.7) and (3.8) with the boundary conditions (3.9), whereas lemma 1 and theorem 1 in themselves are sufficient to disprove the existence of solutions which are physically plausible.

6. Physical interpretation of the results

To what extent does one have fluid flows with saddle points of attachment which may be quantitatively explained by our results? Also, how useful a guide are our solutions, when it is remembered that we used a mainflow which was a linear function of the co-ordinates, so that our solutions are only applicable in the immediate neighbourhood of the stagnation point?

The possibility of saddle points of attachment occurring in inviscid flows was demonstrated in §2. Consider now a body, symmetrical about the planes $x = 0$, $y = 0$, possessing two protuberances as shown in figure 3. Here the z-axis is normal to the body at Z_3 and the (x, y)-plane is the tangent plane at Z_3, the y-axis lying in the plane of the paper.

When this body is placed in a uniform stream U_∞ parallel to the z-direction there will be nodal points of attachment at the protuberances if the rest of the body is slenderly swept back. If the curve C, the intersection of the surface of the body with the (y, z)-plane is indented only slightly between the nodes, there will be a saddle point of attachment at Z_3. The flow in the neighbourhood of Z_3 will be given by the solution of equation (3.7) and (3.8) for some value of c between 0 and $-0{\cdot}4294$.

In figure 4 the full lines give the mainflow in the y-direction near the nodes and the saddle point, where it depends linearly on y. It is reasonable to suppose

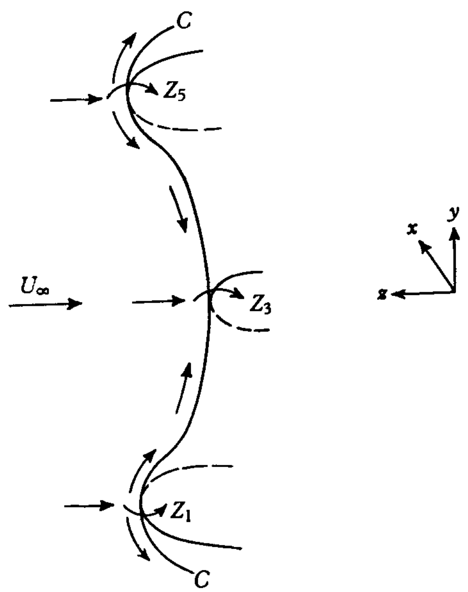

FIGURE 3. Flow past a body with two protuberances.

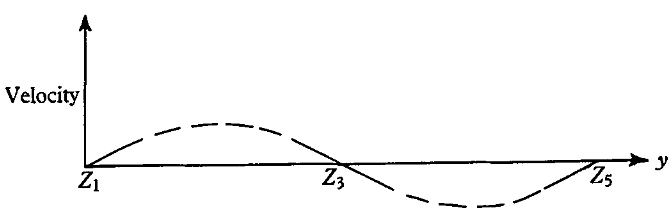

FIGURE 4. The external velocity in the y-direction.

that between these points the mainflow may be extrapolated via the dashed lines. These would not cross the axis, for if so, we would have more nodes and saddles but we are explicitly considering a saddle point of the flow and its two adjacent nodes. In this case, where the flow near Z_3 is given by equations (3.7) and (3.8) for some value of c between 0 and $-0{\cdot}4294$, the pattern of the skin-friction lines on the surface of the body is as shown in figure 5.

Two cases are possible according as to which direction the skin-friction lines at Z_1 and Z_5 are tangential. A general guide is that they will be tangential to the direction of greatest principal curvature at these points, which direction will depend on the shape of the protuberances. The topography for the two cases is, however, essentially unchanged save near these points, and we illustrate the case when the protuberances extend more in the x-direction than in the y-direction.

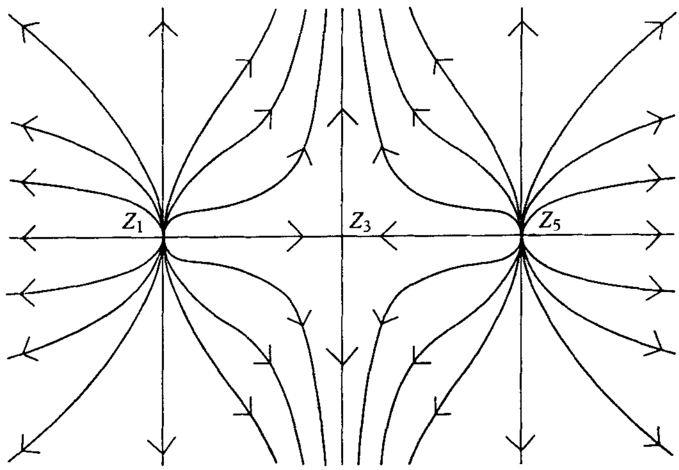

FIGURE 5. The pattern of skin-friction lines when the flow near Z_3 is given by equations (3.7) and (3.8) for a value of $c > -0{\cdot}4294$.

Now suppose we bend the curve C in figure 3 further inwards at Z_3 so as to increase the inflow in the y-direction towards Z_3. Then the corresponding value of c, appropriate to the flow near Z_3, will diminish and if the inflow towards Z_3 is great enough the flow would reverse its direction along and near to the y-axis. It is possible that if this happened the backflow would spread causing additional saddle points of attachment Z_2 and Z_4 (at which there was no reversed flow) in the flow very close to the surface and in the skin-friction lines, the original saddle point having changed into a node so that the new pattern is as in figure 6. Note that in the region shown here the excess number of nodal points over saddle points is unchanged as required (see Appendix).

Since streamlines from far upstream now attach themselves at Z_2 and Z_4, the surface flow near Z_3 is considerably altered, however, the general structure of the flow is readily inferred on examining figure 6. The lines SS through Z_2 and Z_4 will be lines of separation of limiting streamlines and the line AA through Z_3 will be a line of attachment. One infers that fluid separates, in the form of bubbles from the lines SS, the surfaces of separation curling over as fluid falls down the 'back' of the saddle and attaching themselves to the line AA. This means that mainstream fluid which approaches the body between the streamline surfaces through say the stagnation streamlines which attach themselves to Z_2 and Z_3, falls off the 'back' of the saddle, rotates, and stretched away like a vortex in a clockwise direction viewed from figure 7. Similarly another vortex rolls away on the other side of Z_3 in an anti-clockwise direction.

The presence of the vortex bubble may invalidate the assumption that the flow near Z_3 is of boundary-layer nature, and so solutions of equations (3.7) and (3.8) for values of c less than $-0{\cdot}4294$ may be only of theoretical interest. When one has flow over a semi-infinite plane, the only singularity being the saddle point, backflow may be said to occur where

$$\mathbf{v}_e \cdot \mathbf{v}_s < 0. \tag{6.1}$$

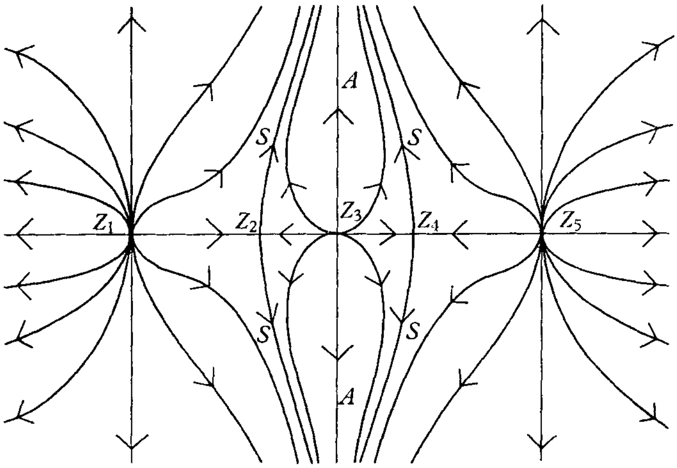

FIGURE 6. The pattern of skin-friction lines when reverse flow occurs in the boundary layer near Z_3.

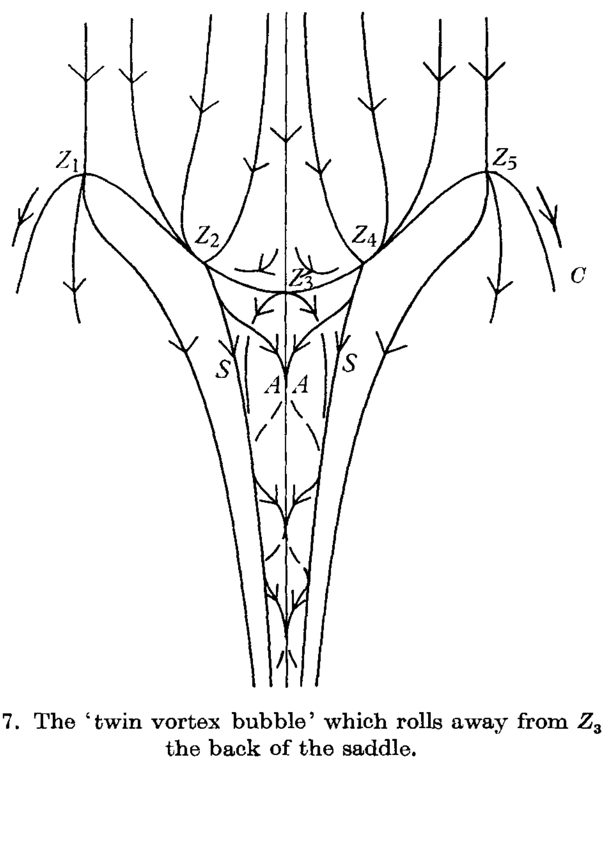

FIGURE 7. The 'twin vortex bubble' which rolls away from Z_3 down the back of the saddle.

Here v_e is the external mainflow and v_s is the velocity within the boundary layer. Near the saddle point Z_3 then there is backflow where

$$a^2x^2f'(\eta)+b^2y^2g'(\eta)+\frac{\nu}{a}(a+b)\,\eta\{af(\eta)+bg(\eta)\} < 0. \tag{6.2}$$

For fixed η this is the region contained by a hyperbola whose asymptotes L_1, L_2 are the lines

$$a^2x^2f'(\eta)+b^2y^2g'(\eta) = 0. \tag{6.3}$$

Velocities normal to the surface $\eta = 0$ are small even for a moderate Reynolds number, so that such hyperbolae lie close to their asymptotes. In fact as $\eta \to 0$ the associated hyperbola reduces to its asymptotes which then pass through the origin. For fixed η the region of backflow diminishes as η increases until it becomes zero at $\eta = \bar{\eta}$, where $g'(\eta) = 0$ again, when the two lines L_1, L_2 coincide.

7. Numerical results

Solutions of equations (3.7) and (3.8) subject to the boundary conditions (3.9) were obtained for thirteen negative values of c and in particular for $c = 0$, $-0{\cdot}25$, $-0{\cdot}4294$, $-0{\cdot}50$, $-0{\cdot}75$ and -1. The initial values $f''(0)$ and $g''(0)$, together with the three boundary-layer displacement thicknesses, are given in

c	$f''(0)$	$g''(0)$	$\frac{a^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\delta_1$	$\frac{a^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\delta_x$	$\frac{a^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\delta_y$
0·0	1·2326	0·5705	0·648	0·648	1·026
−0·1	1·2284	0·4594	0·600	0·654	1·141
−0·2	1·2258	0·3353	0·501	0·658	1·287
−0·25	1·2251	0·2680	0·421	0·659	1·375
−0·3	1·2250	0·1970	0·311	0·660	1·474
−0·4	1·2265	0·0460	−0·036	0·659	1·702
−0·4294	1·2273	0	−0·183	0·658	1·776
−0·5	1·2302	−0·1115	−0·652	0·655	1·962
−0·6	1·2359	−0·2666	−1·726	0·649	2·232
−0·7	1·2432	−0·4130	−3·688	0·641	2·497
−0·75	1·2473	−0·4821	−5·330	0·637	2·625
−0·8	1·2517	−0·5488	−7·852	0·632	2·753
−0·9	1·2612	−0·6761	−20·96	0·622	3·018
−1·0	1·2729	−0·8112	−∞	0·610	3·438

TABLE 1

table 1 for all thirteen cases considered and the corresponding quantities for $c = 0$ are also given for comparison. Quantities in this table are correct to the number of decimal places given. One may note from this table that δ_1, the three-dimensional displacement thickness, is negative for a range of c for which reverse flow does not occur. As c approaches -1 then δ_1 becomes large and negative. When $c = -1$ we have $\delta_1 = -\infty$ and $\delta_y - \delta_x = 2{\cdot}828$. This last figure was evaluated more accurately and found to be equal to $2\sqrt{2}$ to five decimal places.

Table 2 contains, for the six values of c mentioned in the text, the functions f' and g' correct to three decimal places.†

As Howarth found for his range of c, it is also the case here that for $-1 \leqslant c \leqslant 0$ changes with c are small for the velocity in the x-direction, though much greater in the y-direction, the direction in which backflow occurs. This is indicated by the growing difference between δ_x and δ_y as c becomes more negative. For negative values of c, $f''(0)$ reaches a minimum value of 1·225 near $c = -0{\cdot}3$ and then rises to 1·273 when $c = -1$ so that over the whole range from $c = 1$, when $f''(0) = 1{\cdot}312$, to $c = -1$, $f''(0)$ varies very little. However $g''(0)$, which is also

	$c = 0$		$c = -0{\cdot}25$		$c = -0{\cdot}4294$	
η	f'	g'	f'	g'	f'	g'
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·1	0·118	0·057	0·118	0·028	0·118	0·002
0·2	0·227	0·114	0·225	0·059	0·226	0·009
0·3	0·325	0·171	0·323	0·092	0·324	0·019
0·4	0·414	0·227	0·412	0·127	0·412	0·034
0·5	0·495	0·284	0·491	0·164	0·492	0·053
0·6	0·566	0·339	0·562	0·204	0·563	0·076
0·7	0·630	0·393	0·625	0·245	0·626	0·103
0·8	0·686	0·446	0·681	0·287	0·682	0·134
0·9	0·735	0·498	0·729	0·331	0·731	0·167
1·0	0·778	0·547	0·772	0·375	0·773	0·204
1·1	0·815	0·594	0·809	0·419	0·810	0·243
1·2	0·847	0·639	0·841	0·464	0·841	0·284
1·3	0·874	0·681	0·868	0·508	0·868	0·327
1·4	0·897	0·720	0·891	0·551	0·891	0·371
1·5	0·916	0·756	0·910	0·593	0·911	0·415
1·6	0·932	0·789	0·927	0·633	0·927	0·460
1·7	0·946	0·819	0·941	0·672	0·941	0·504
1·8	0·957	0·846	0·952	0·708	0·952	0·547
1·9	0·966	0·870	0·961	0·742	0·961	0·589
2·0	0·973	0·891	0·969	0·774	0·969	0·629
2·2	0·984	0·926	0·981	0·829	0·980	0·703
2·4	0·991	0·951	0·988	0·875	0·988	0·769
2·6	0·995	0·969	0·993	0·911	0·993	0·824
2·8	0·997	0·981	0·996	0·938	0·996	0·869
3·0	0·998	0·989	0·998	0·958	0·997	0·905
3·2	0·999	0·993	0·999	0·973	0·998	0·933
3·4	1·000	0·996	0·999	0·983	0·999	0·954
3·6	—	0·998	1·000	0·989	1·000	0·969
3·8	—	0·999	—	0·994	—	0·979
4·0	—	1·000	—	0·996	—	0·987
4·5	—	—	—	0·999	—	0·996
5·0	—	—	—	1·000	—	0·999
6·0	—	—	—	—	—	1·000

TABLE 2

† A table of values of f and g and their first and second derivatives for these same values of c has been lodged with the Editor of the *Journal of Fluid Mechanics* and may be consulted by readers on application to the Editor.

η	$c = -0{\cdot}50$ f'	g'	$c = -0{\cdot}75$ f'	g'	$c = -1$ f'	g'
0·0	0·000	0·000	0·000	0·000	0·000	0·000
0·1	0·118	−0·009	0·120	−0·044	0·122	−0·076
0·2	0·226	−0·012	0·230	−0·081	0·235	−0·142
0·3	0·325	−0·011	0·330	−0·111	0·337	−0·198
0·4	0·413	−0·005	0·420	−0·133	0·430	−0·244
0·5	0·493	0·007	0·501	−0·147	0·513	−0·280
0·6	0·565	0·023	0·574	−0·154	0·588	−0·307
0·7	0·628	0·043	0·638	−0·153	0·653	−0·323
0·8	0·684	0·069	0·695	−0·145	0·711	−0·330
0·9	0·733	0·098	0·744	−0·130	0·760	−0·328
1·0	0·775	0·131	0·786	−0·109	0·803	−0·318
1·1	0·812	0·167	0·823	−0·082	0·840	−0·299
1·2	0·843	0·207	0·854	−0·049	0·870	−0·273
1·3	0·870	0·248	0·881	−0·011	0·896	−0·241
1·4	0·893	0·292	0·903	0·031	0·917	−0·203
1·5	0·912	0·337	0·922	0·076	0·935	−0·160
1·6	0·928	0·382	0·937	0·124	0·949	−0·114
1·7	0·942	0·428	0·950	0·175	0·960	−0·064
1·8	0·953	0·473	0·960	0·226	0·969	−0·012
1·9	0·962	0·518	0·968	0·279	0·976	0·042
2·0	0·970	0·562	0·975	0·331	0·982	0·097
2·2	0·981	0·644	0·985	0·433	0·990	0·206
2·4	0·988	0·717	0·991	0·529	0·994	0·311
2·6	0·993	0·780	0·995	0·617	0·997	0·410
2·8	0·996	0·833	0·997	0·694	0·998	0·500
3·0	0·997	0·876	0·998	0·759	0·999	0·581
3·2	0·999	0·911	0·999	0·814	1·000	0·651
3·4	0·999	0·937	0·999	0·859	—	0·712
3·6	1·000	0·956	1·000	0·895	—	0·764
3·8	—	0·970	—	0·923	—	0·807
4·0	—	0·980	—	0·944	—	0·844
4·5	—	0·994	—	0·977	—	0·909
5·0	—	0·998	—	0·991	—	0·948
6·0	—	1·000	—	0·999	—	0·984
7·0	—	—	—	1·000	—	0·995
9·0	—	—	—	—	—	1·000

TABLE 2 (*cont.*)

1·312 when $c = 1$, is zero when $c = -0{\cdot}4294$ and becomes negative attaining a minimum value of $-0{\cdot}811$ when $c = -1$. The function $\bar{\eta} = \bar{\eta}(c)$ mentioned at the end of the previous section, which measures, when $c < -0{\cdot}4294$, the height of the back-flow region, has a maximum value of 1·82 when $c = -1$.

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Appendix: singular points

If, as in the Introduction, we have viscous fluid flowing past a body which is topologically equivalent to a sphere and which has a smooth surface, the fluid velocity at a small distance z from the surface is $\boldsymbol{\epsilon} z + O(z^2)$, where $\boldsymbol{\epsilon}$, a function of position of a point P on the surface S, lies in the tangent plane to S at P. Let $\mathbf{n}$ denote the unit vector along the outward normal to S at P and let $\boldsymbol{\omega}$ denote the fluid vorticity in the tangent plane to S at P so that $\boldsymbol{\epsilon} = \boldsymbol{\omega} \wedge \mathbf{n}$. Then the system of lines $\boldsymbol{\epsilon}$ parallel to $\mathbf{dr}$ and $\boldsymbol{\omega}$ parallel to $\mathbf{dr}$, where $\mathbf{dr}$ is the relative position of two adjacent points on a curve, form an orthogonal net on S, the latter being the vortex lines on the surface.

A point of S at which $\boldsymbol{\epsilon} = \boldsymbol{\omega} = 0$ is a stagnation point of the flow and is a singular point of the differential equations of both systems of curves which map S. Let P be such a singular point and take rectangular Cartesian co-ordinates $P(x, y, z)$ such that Pz is along the outward normal to the surface and $P(x, y)$ is the tangent plane to S at P.

By continuity of the flow the normal component of velocity w is given by

$$w = -\tfrac{1}{2}\Delta z^2 + O(z^3),$$

where $\Delta = \operatorname{div} \boldsymbol{\epsilon}$ is the two-dimensional divergence of $\boldsymbol{\epsilon}$.

According as to whether $\Delta > 0$ or $\Delta < 0$ we say that P is a stagnation point of attachment or separation, respectively.

Now suppose $\boldsymbol{\epsilon}$ has components ϵ_x, ϵ_y with respect to the x, y-axes and define

$$J = \frac{\partial \epsilon_x}{\partial x}\frac{\partial \epsilon_y}{\partial y} - \frac{\partial \epsilon_x}{\partial y}\frac{\partial \epsilon_y}{\partial x}.$$

J is invariant with respect to a rotation about P of the x, y-axes in the tangent plane at P and according as $J > 0$ or $J < 0$ we say that P is a nodal point or a saddle point, respectively.

Also, the vorticity normal to the surface is

$$\left(\frac{\partial \epsilon_y}{\partial x} - \frac{\partial \epsilon_x}{\partial y}\right) z + O(z^2),$$

and at a stagnation point of attachment of a streamline from far upstream where the flow is irrotational we must have

$$\frac{\partial \epsilon_y}{\partial x} - \frac{\partial \epsilon_x}{\partial y} = 0,$$

since fluid very close to the stagnation streamline can in no way acquire vorticity. This means that by suitably choosing our x- and y-axis $\boldsymbol{\epsilon}$ takes to first order in x and y the form (ax, by) near P.

This is the case which covers the stagnation point flows discussed in the present paper. For this case we have $\Delta = a + b$ and $J = ab$. Since $\Delta > 0$, one of a, b must be positive and we choose the greater to be $a > 0$. Hence $J = a^2(b/a)$, so that for $b > 0$ we have a nodal point and for $b < 0$ we have a saddle point.

We now give a proof that the number of nodal points on S exceeds the number of saddle points by 2. We assume that the skin-friction lines on S have a finite number of distinct nodes and saddles at which $\boldsymbol{\epsilon} = 0$, where we say the vector field has an isolated singularity.

Let A be a non-intersecting polygon on S whose sides $C_1, C_2 \ldots, C_n$ are geodesics which do not pass through a singularity of $\boldsymbol{\epsilon}$. Define an integer

$$\Delta(A) = \frac{1}{2\pi}\left(\sum_i [\theta]_{C_i} + \int_A K\,d\sigma\right),$$

where $[\theta]_{C_i}$ is the variation along C_i in a counter-clockwise direction of the angle θ between the tangent $\mathbf{t}$ to C_i and the vector $\boldsymbol{\epsilon}$, and K is the Gaussian curvature of S.

In the case of a small (almost plane) polygon enclosing a singularity of $\boldsymbol{\epsilon}$, $\int_A K\,d\sigma \doteqdot 0$ and $\Delta(A)$ coincides with the index of the singularity, defined to be the variation of θ round the polygon.

In general $\int_A K\,d\sigma$ is a correction term to make $\Delta(A)$ an integer, and is the measure of the non-Euclidean nature of S, for

$$\int_A K\,d\sigma = 2\pi - \sum (\text{exterior angles of } A);$$

this is the content of the Gauss–Bonnet Theorem (Eisenhart 1940). It is easily shown that if A, B are polygons overlapping only on their edges, then

$$\Delta(A) + \Delta(B) = \Delta(A+B),$$

also if A is small and contains at most one singularity of $\boldsymbol{\epsilon}$, then $\Delta(A) = 1, -1, 0$ according as A contains a node, a saddle point, or no singularity of $\boldsymbol{\epsilon}$. Thus for any A, $\Delta(A) = n_A - s_A$ where A contains n_A nodes and s_A saddle points of $\boldsymbol{\epsilon}$.

Thus if S contains n nodes and s saddle points, then

$$n - s = \Delta(A) + \Delta(S-A) = \frac{1}{2\pi}\int_S K\,d\sigma,$$

since the terms $[\theta]_{C_i}$ cancel in the sum. The expression on the right is the Euler–Poincaré characteristic of S, and is 2 for a sphere. Thus we have for S that $n - s = 2$ as required.

If, for instance, $s = 0$ there will be just one nodal point of attachment on the body. The surface streamlines will diverge therefrom and then converge to leave the body at one nodal point of separation, If, however, there are two nodal points of attachment the surface streamlines from each must divide somewhere at a saddle point of the surface flow. Similarly, if one has two nodal points of separation these will also have associated with them a saddle point, the streamlines from which will diverge to one or other of these nodal points of separation. Thus, over and above the two original nodal points, one of attachment and one of separation, for each additional nodal point there will be an additional saddle point so that $n - s = 2$ in all possible cases.

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